Reflections on the Liquid Tensor Experiment

by *Johan Commelin*

Liquid Tensor Experiment

On 5 Dec 2020, Peter Scholze posted a challenge:

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Check the main theorem of liquid vector spaces

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…on a computer

1. Condensed/liquid mathematics
2. The experiment
3. Reflections

Condensed mathematics

Developed by Dustin Clausen and Peter Scholze.

New theory of algebraic objects with a topological flavour such as topological groups/rings/vector spaces

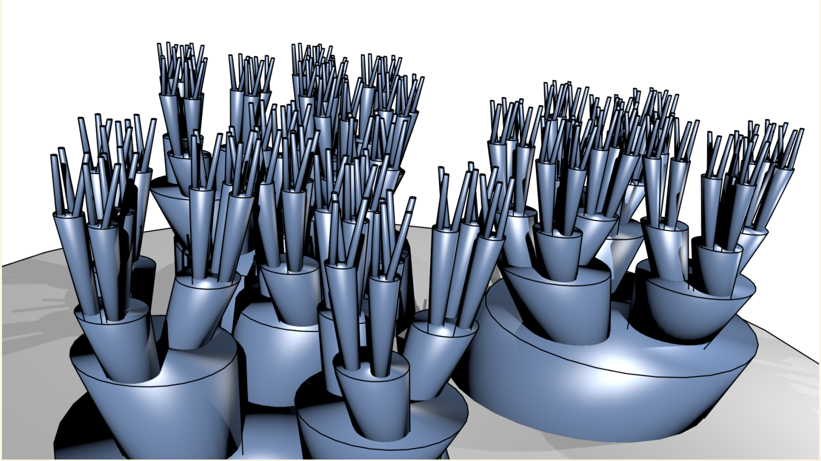
Makes homological algebra applicable in this context

A continuous problem

$$\mathbb{R}^{\delta} \to \mathbb{R}$$

Unifying geometries

- real/complex geometry
- algebraic geometry
- p-adic geometry



3-adics, by Daniel Litt

Entering the condensed world

$$\begin{aligned} \mathrm{Cond}(\mathrm{Set}) &= \{\text{“sheaves on profinite sets”}\} \\ \mathrm{CompHaus} &\hookrightarrow \mathrm{Cond}(\mathrm{Set}) \\ \text{“functional analysis”} &\hookrightarrow \mathrm{Cond}(\mathbb{R}\text{-Vect}) \end{aligned}$$

“Liquid vector spaces” = abelian subcategory of Cond($\mathbb{R}$-Vect) with good tensor product.

Theorem (Clausen–Scholze)

Let $0 < p' < p < 1$ be real numbers,
let S be a profinite set,
and let V be a p-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S.

Then

$$\operatorname{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geqslant 1$.

Ingredients in the proof

the real numbers, binary expansions of reals, normed groups, p-Banach spaces, locally constant functions, completions (of metric spaces), absolute convergence and infinite sums, pseudo-normed groups, abelian categories, homological complexes, homotopies, the snake lemma, profinite sets, Čech covers, the normalized Moore complex, polyhedral lattices, Gordan's lemma, derived functors, long exact sequences, more homological algebra, condensed sets, extremally disconnected sets, sheaves, more topos theory, condensed abelian groups, some form of Breen–Deligne resolutions

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Scholze (LTE blogpost, 2020)

- “spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it”
- “proof has some very unexpected features … very much of arithmetic nature”
- “I think this may be my most important theorem to date”
- “I think nobody else has dared to look at the details, and so I still have some small lingering doubts”

Progress report

target 1 100%

target 2 90%

- Main technical theorem verified in half a year
- Some statements/proofs of lemmas and auxiliary definitions were changed
- Detailed blueprint
- Answer to Question 9.9 of `Analytic.pdf`
- Proof almost too big for human brain, not for Lean

Scholze (LTE blogpost, 2021)

“*Question: Interesting! What else did you learn?*

Answer: What actually makes the proof work!

[…] this made me realize that actually the key thing happening is a reduction from a non-convex problem over the reals to a convex problem over the integers.”

Breen–Deligne resolutions

As part of the project, I came up with an alternative to Breen–Deligne resolutions.

The new method is easier to prove and easier to apply.

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Caveat emptor

- I'm a *type pragmatist* not a type theorist.
- These reflections have not been formally verified!

Blueprint

- We used a blueprint, but it was co-developed with the formalization.
- So not really a blueprint.
- It gave a very clear overview of the state of the project. Especially the dependency graph.

Refactors

- ITP’s are powerful.
- The cost of a refactor is much lower than in traditional software.
- Don’t be scared of them.

Top-down

- Great way of figuring out correct lower-level concepts and definitions.
- Only if you know where you're heading.
- If you mess up, you are faced with a (huge?) refactor.
- Might discover new maths! (Breen–Deligne)

Bottom-up

- If you don’t know where you are heading.
- Just battle along, until you’ve figured it out.
- Might still imply refactor, or fix things with glue.
- (Duct tape maths!?)

Commutative diagrams

- Diagrams are 2D. Lean is 1D.
- Current situation is cumbersome. Doesn't scale.
- Ripe for automation: "Just follow your nose"

Definitional equality matters

- More than it should.
- No distinction between defeq and propeq in "ordinary maths".
- But DTT is really great in general!
- Some sort of extensional type theories maybe?

Reasoning up to isomorphism

- Happens all over the place in LTE.
- And in algebraic geometry/topology more generally.
- We'll probably need something like HoTT.
- But do we want it everywhere? Or just for the categorical parts of formalization?
- Maybe as a meta-theoretical tool?
- I'm rooting for all the people working on foundational issues.

Slow-downs

- Complexity leads to slowdowns.
- Stict APIs and irreducible defs are one mitigation.
- Round-tripping pretty-printer is powerful.
- Automation can be unbearably slow (`tidy` and `simps`)
- Other factors? Other solutions?

Response of the maths community

- Formalization/ATP/machine-maths is now considered a “hot topic” in the maths community.

Thank you!